RATIONAL FUNCTIONS IN NONCOMMUTATING VARIABLES, THEIR REALIZATIONS, AND APPLICATIONS TO LINEAR MATRIX INEQUALITIES

Victor Vinnikov (Ben Gurion University)

(mostly based on joint work with Bill Helton (UCSD) and Scott McCullough (University of Florida), and with Dmitry Kaliuzhnyi-Verbovetskyi (Drexel University)

What are noncommutative rational functions?

Rational functions in several *noncommuting* variables (as opposed to the "usual" rational functions in several *commuting* variables).

Examples:

- $z_1 z_2$
- $z_2 z_1$
- $1 + (z_1^{-1} z_2 + z_2^{-1} z_1)^{-1}$
- $z_1 z_2 (z_1 z_2 - z_2 z_1)^{-1} = 1 + z_2 z_1 (z_1 z_2 - z_2 z_1)^{-1}$

Some difficulties:

- Unlike in the commutative case, the "minimal complexity" of an expression defining a given noncommutative rational function can be arbitrarily high; there is nothing similar to a coprime fraction representation.
- It is not clear when two different expressions define the same noncommutative rational function

This reflects a familiar difficulty in noncommutative algebra: constructing a skew field of fractions of a noncommutative integral domain (in this case, the domain of noncommutative polynomials) is a non-trivial issue (a skew field of fractions does not necessarily exist, and when it does, it is not necessarily unique).

However, these difficulties can be resolved, and *in many instances noncommutative rational functions behave in a much simpler way than rational functions in several commuting variables* (much more like rational functions in a single variable). Furthermore, *noncommutative rational functions occur in many areas of system theory*:

- First appearence in the context of rational and recognizable formal power series in noncommuting variables in the theory of formal languages and finite automata (Kleene, Schützenberger, Fliess).
- More recently, state space realizations of rational expressions in Hilbert space operators (modelling structured possibly time varying uncertainty) have figured prominently in work on robust control of linear systems (Beck, Beck–Doyle–Glover, Lu–Zhou–Doyle).
- Another important application: Linear Matrix Inequalities (LMIs) in the context of dimension-independent problems, i.e., where the natural variables are matrices and the problem involves rational expressions in these matrix variables.
- Last but not least, in many situations one can establish a commutative result by "lifting" to the noncommutative setting, applying the noncommutative theory, and then "descending" again to the commutative situation.

Purpose of this talk: publicity for noncommutative rational functions.

More specifically, I will

(1) survey the basic concepts of the theory of noncommutative rational functions,

(2) their realization theory, and

(3) their applications to LMIs.

Important remark: the NCAlgebra software,

http://www.math.ucsd.edu/ ∼ ncalg,

implements many symbolic algorithms in the noncommutative setting.

We start with noncommutative polynomials in d noncommuting variables $z_1, \ldots, z_d$ over the field $\mathbb{R}$ of real numbers ($\mathbb{R}$ can be replaced for most purposes by any field $\mathbb{K}$, though a little bit of care is necessary, especially in case of finite fields).

Example: a noncommutative polynomial of total degree 2 in 2 variables z_1, z_2 is of the form

$$a + bz_1 + cz_2 + dz_1^2 + ez_1z_2 + fz_2z_1 + gz_2^2,$$

where the coefficients $a, b, c, d, e, f, g \in \mathbb{R}$.

Noncommutative polynomials form an algebra $\mathbb{R}\langle z_1, \ldots, z_d\rangle$ over $\mathbb{R}$, often called the free associative algebra on d generators $z_1, \ldots, z_d$.

We can evaluate a noncommutative polynomial $p \in \mathbb{R}\langle z_1, \ldots, z_d\rangle$ on a d-tuple $Z_1, \ldots, Z_d$ of $n \times n$ matrices over $\mathbb{R}$, for any n, yielding a $n \times n$ matrix $p(Z_1, \ldots, Z_d)$. A non-zero polynomial can vanish on tuples of matrices of a certain size; e.g., $z_1z_2 - z_2z_1$ vanishes on 1×1 matrices (scalars). However, if $p(Z_1, \ldots, Z_d) = 0$ for all d tuples of matrices of all sizes, then necessarily p is the zero polynomial.

We next define noncommutative rational expressions by applying successive arithmetic operations to noncommutative polynomials. A noncommutative rational expression r can be evaluated on a d-tuple $Z_1, \ldots, Z_d$ of $n \times n$ matrices in its *domain of regularity* $\operatorname{dom} r$, which is defined as the set of all d-tuples of matrices of all sizes such that all the inversions involved in the calculation of $r(Z_1, \ldots, Z_d)$ exist.

Example: if $r = (z_1 z_2 - z_2 z_1)^{-1}$ then $\operatorname{dom} r = \{(Z_1, Z_2) : \det(Z_1 Z_2 - Z_2 Z_1) \neq 0\}$.

We assume that $\operatorname{dom} r \neq \emptyset$, in other words, when forming noncommutative rational expressions we never invert an expression that is nowhere invertible.

Two noncommutative rational expressions r_1 and r_2 are called *equivalent* if $\operatorname{dom} r_1 \cap \operatorname{dom} r_2 \neq \emptyset$ and $r_1(Z_1, \ldots, Z_d) = r_2(Z_1, \ldots, Z_d)$ for all $(Z_1, \ldots, Z_d) \in \operatorname{dom} r_1 \cap \operatorname{dom} r_2$.

Example: $r_1 = z_1 z_2 (z_1 z_2 - z_2 z_1)^{-1}$ and $r_2 = 1 + z_2 z_1 (z_1 z_2 - z_2 z_1)^{-1}$ are equivalent.

We define a *noncommutative rational function* to be an equivalence class of noncommutative rational expressions. We usually denote noncommutative rational functions by German (Fraktur) letters; we define the *domain of regularity* of a noncommutative rational function $\mathfrak{r}$ as the union of the domains of regularity of all noncommutative rational expressions representing this function, i.e.,

$$\operatorname{dom} \mathfrak{r} = \cup_{r \in \mathfrak{r}} \operatorname{dom} r.$$

Notice that for any d-tuple $(Z_1, \ldots, Z_d) \in \operatorname{dom} \mathfrak{r}$ of $n \times n$ matrices, the evaluation $\mathfrak{r}(Z_1, \ldots, Z_d)$ is a well defined $n \times n$ matrix.

Any nonzero noncommutative rational function is invertible; i.e., noncommutative rational functions form a skew field — a skew field of fractions (in fact, the universal skew field of fractions) of the ring of noncommutative polynomials. (This means that, e.g., if p is a noncommutative polynomial and $\det p(Z_1, \ldots, Z_d) = 0$ for all d tuples of matrices of all sizes, then necessarily p is the zero polynomial.)

We also introduce matrix-valued noncommutative rational expressions and matrix-valued noncommutative rational functions. The only difference is that we start with matrix-valued noncommutative polynomials (having matrix rather than scalar coefficients) and use tensor substitutions for evaluations on tuples of matrices.

Example:

$$r_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 - z_1 & -z_2 \\ -z_2 & 1 - z_1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$r_2 = (1 - z_1 - z_2(1 - z_1)^{-1} z_2)^{-1},$$

and

$$r_3 = -z_2^{-1}(1 - z_1)(z_2 - (1 - z_1) z_2^{-1}(1 - z_1))^{-1}$$

are equivalent, and we have

$$\operatorname{dom} r_1 = \left\{ (Z_1, Z_2) : \det \begin{bmatrix} 1 - Z_1 & -Z_2 \\ -Z_2 & 1 - Z_1 \end{bmatrix} \neq 0 \right\},$$

$$\begin{aligned} \operatorname{dom} r_2 = \{(Z_1, Z_2) : \det(1 - Z_1) \neq 0, \\ \det(1 - Z_1 - Z_2(1 - Z_1)^{-1} Z_2) \neq 0\}, \end{aligned}$$

$$\begin{aligned} \operatorname{dom} r_3 = \{(Z_1, Z_2) :\ \det(Z_2) \neq 0, \\ \det(Z_2 - (1 - Z_1) Z_2^{-1}(1 - Z_1)) \neq 0\}. \end{aligned}$$

A noncommutative multidimensional system is a system with evolution along the free semigroup $\mathcal{F}_d$ on d letters $g_1, \dots, g_d$ rather than along the multidimensional integer lattice $\mathbb{Z}^d$. An example of system equations with evolution along $\mathcal{F}_d$ is given by

$$\left\{ \begin{array}{rcl} x(g_1 w) & = & A_1 x(w) + B_1 u(w) \\ & \vdots & \\ x(g_d w) & = & A_d x(w) + B_d u(w) \\ y(w) & = & Cx(w) + Du(w) \end{array} \right. \tag{1}$$

where the variable $w = g_{i_n} \dots g_{i_1}$ is a word in the symbols $g_1, \dots, g_d$, i.e., a generic element of the free semigroup $\mathcal{F}_d$.

Applying to the system equations (1) an appropriately defined formal noncommutative z-transform and under the assumption that the state of the system is initialized at 0, we arrive at the input-output relation

$$\widehat{y}(z) = T_\Sigma(z)\widehat{u}(z)$$

where the *transfer function* is given by

$$T_\Sigma(z) = D + C(I - A_1 z_1 - \cdots - A_d z_d)^{-1}(B_1 z_1 + \cdots + B_d z_d). \tag{2}$$

We see that the transfer function is a matrix-valued noncommutative rational function in noncommuting variables $z_1, \dots, z_d$ which is regular at zero, i.e., zero belongs to its domain of regularity (a little more precisely, the transfer function is the matrix-valued noncommutative rational function defined by the matrix-valued noncommutative rational expression (2)).

Foundational facts of the noncommutative realization theory:

(1) Every matrix-valued noncommutative rational function which is regular at zero admits a state space realization (2).

(2) An arbitrary realization of a given matrix-valued noncommutative rational function can be reduced via an analogue of the Kalman decomposition to a controllable and observable realization.

(3) A controllable and observable realization is minimal, i.e., it has the smallest possible state space dimension, and is unique up to a unique similarity.

(4) A minimal realization can be constructed canonically and explicitly from a matrix-valued noncommutative rational function by means of the corresponding Hankel operator; this ties in with the fact that the Hankel operator corresponding to a matrix-valued noncommutative formal power series has finite rank if and only if the power series represents a rational function (an analogue of Kronecker's Theorem).

(5) In a minimal realization, the singularities of the transfer function coincide with the singularities of the resolvent; more precisely, the domain of regularity of the transfer function is exactly

$$\{(Z_1, \ldots, Z_d)\colon \det(I - A_1 \otimes Z_1 - \cdots - A_d \otimes Z_d) \neq 0\}.$$

Items 1–4 are due to Ball–Groenewald–Malakorn, in a more general setting of structured noncommutative multidimensional systems (they go back to Kleene, Schützenberger, and Fliess in the setting of recognizable formal power series).

Item 5 is amazingly difficult to prove “by hands”; the classical $d = 1$ proof uses the Jordan canonical form for $A = A_1$, but this is of course no longer available. A recent proof by Kaliuzhnyi-Verbovetskyi–Vinnikov uses noncommutative backward shifts which are a particular instance of a difference-differential calculus for noncommutative rational functions. This is a special case of a difference-differential calculus for general noncommutative functions, which are functions on tuples of matrices of all sizes which respect direct sums and simultaneous similarities.

Most optimization problems appearing in systems and control are *dimension-independent* in that the natural variables are matrices (rather than just collections of scalars) and the problem involves rational expressions in these matrix variables (rather than arbitrary expressions in the matrix entries).

In a dimension-independent problem, we are given a convex set $\mathcal{S}_n$ of d-tuples of $n \times n$ real symmetric matrices for each n. We seek a noncommutative LMI representation of the collection $\mathcal{S} = \{\mathcal{S}_n\}_{n=1}^{\infty}$, i.e., an affine linear matrix-valued noncommutative polynomial

$$L = L_0 + L_1 z_1 + \cdots + L_d z_d \tag{3}$$

with real symmetric coefficients such that

$$\mathcal{S}_n = \{(Z_1, \ldots, Z_d) \in (\mathbb{SR}^{n\times n})^d \colon L(Z_1, \ldots, Z_d) \succeq 0\}$$

for all n. Here $\mathbb{SR}^{n\times n}$ denotes $n \times n$ real symmetric matrices, and $A \succeq B$ means that $A - B$ is positive semidefinite.

The sets $\mathcal{S}_n$ are supposed to be defined by some matrix-valued noncommutative polynomial or rational inequalities (the same inequalities for all n, i.e., we are evaluating the same matrix-valued noncommutative polynomials or rational functions on matrices of different sizes), and a rough conjecture is that in this noncommutative setting LMI representations always exist.

Strong evidence (Helton–McCullough–Vinnikov): sublevel sets of symmetric noncommutative rational functions that are convex near zero admit noncommutative LMI representations.

More precisely, let $\mathfrak{r}$ be a noncommutative rational function which is regular at zero. Assume that $\mathfrak{r}$ is symmetric (meaning that $\mathfrak{r}(Z_1, \ldots, Z_d)$ is symmetric for all symmetric d-tuples $(Z_1, \ldots, Z_d) \in \operatorname{dom} \mathfrak{r}$) and convex near zero (meaning that

$$\mathfrak{r}(tX_1 + (1-t)Y_1, \ldots, tX_d + (1-t)Y_d) \preceq \\ t\mathfrak{r}(X_1, \ldots, X_d) + (1-t)\mathfrak{r}(Y_1, \ldots, Y_d)$$

for all $0 \leq t \leq 1$ and all d-tuples $(X_1, \ldots, X_d)$, $(Y_1, \ldots, Y_d)$ of symmetric matrices of all sizes with $X_1^2 + \cdots + X_d^2, Y_1^2 + \cdots + Y_d^2 \prec \epsilon I$ for some $\epsilon > 0$).

Then (for any $\gamma > \mathfrak{r}(0)$) the collection of connected components of zero of the sets

$$\mathcal{S}_n = \{(Z_1, \ldots, Z_d) \in (\mathbb{SR}^{n \times n})^d \cap \operatorname{dom} \mathfrak{r} \colon \\ \mathfrak{r}(Z_1, \ldots, Z_d) \prec \gamma I_n\}$$

admits a noncommutative LMI representation.

The proof uses state space realizations of noncommutative rational functions in an essential way. In fact, the LMI representation is constructed from the realization of $\mathfrak{r}$.